Comment on $\displaystyle\int \delta(g(x)-g_0)\, f(x)\, dx = \sum_{x \in g^{-1}(g_0)} \frac{f(x)}{|g'(x)|}$

While one can certainly operate formally as though δ is a function to get this formula, such formal operations are not uniformly successful, and, more importantly, do not reveal why something is true (if indeed it is!)

Here is a derivation that is not just a formal manipulation

(1) If $\delta(x)$ were somehow a function that was 0 everywhere except at $x=0$, then

$$\int \delta(g(x)-g_0)\, f(x)\, dx$$

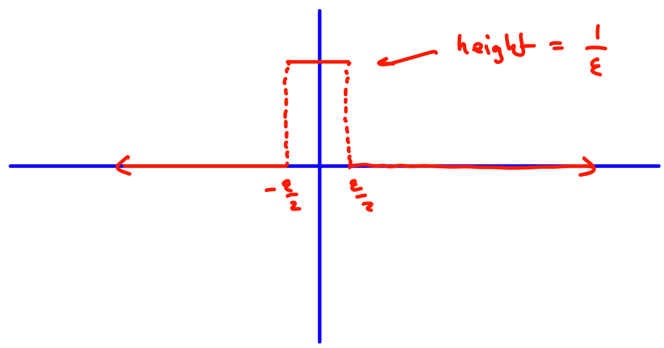
could not know about anything other than the values of g and f at x's where $g(x)=g_0$ ... in particular, because knowing only the value of a function at a point tells you nothing about the derivative of the function at the point this integral could not somehow extract the derivatives of g.

(2) But, actually, $\delta(x)$ is more carefully thought of as either (A) a measure or (B) a linear operator on smooth functions... and implicit in the above integral is the assumption that we are taking the operator approach.

In this case

$$\int \delta(x)\, f(x)\, dx \equiv \lim_{\varepsilon \to 0} \int S_\varepsilon(x)\, f(x)\, dx$$

where $S_\varepsilon(x)$ =



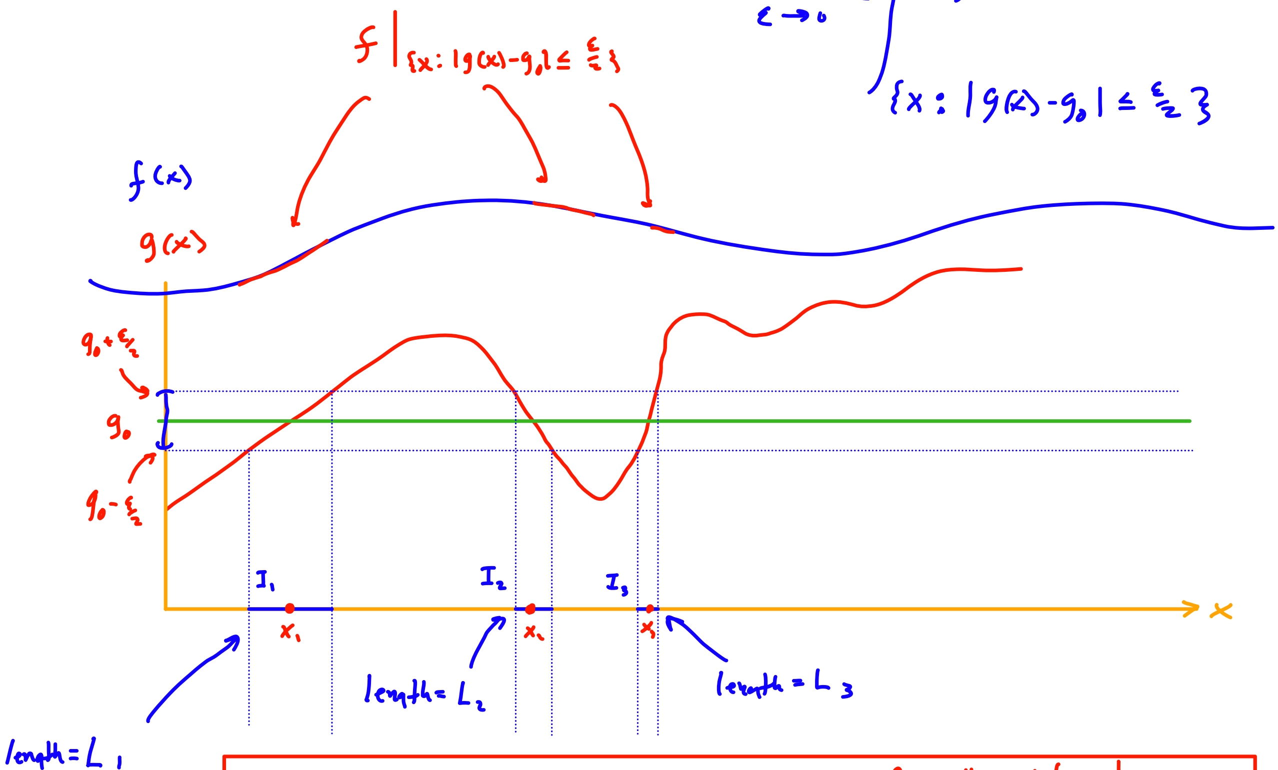
and we immediately get that

$$\int \delta(x)\, f(x)\, dx = f(0)$$

we also immediately get that

$$\int \delta(g(x)-g_0)\, f(x)\, dx \equiv \lim_{\varepsilon \to 0} \int S_\varepsilon(g(x)-g_0)\, f(x)\, dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{x :\, |g(x)-g_0| \le \frac{\varepsilon}{2}\}} f(x)\, dx$$



as $\varepsilon \to 0$ $\quad \frac{\varepsilon}{L_i} \longrightarrow |g'(x_i)|$, $\; i = 1, 2, 3$ and f on those intervals converges to $f(x_i)$

The box immediatly above implies that

$$\frac{1}{\varepsilon}\int_{\{x\,:\,|g(x)-g_0|\le\frac{\varepsilon}{2}\}} f(x)\,dx = \frac{1}{\varepsilon}\int_{I_1} f(x)\,dx + \frac{1}{\varepsilon}\int_{I_2} f(x)\,dx + \frac{1}{\varepsilon}\int_{I_3} f(x)\,dx$$

$$\Rightarrow \frac{L_1}{\varepsilon} f(x_1) + \frac{L_2}{\varepsilon} f(x_2) + \frac{L_3}{\varepsilon} f(x_3)$$

$$\Rightarrow \frac{1}{|g'(x_1)|} f(x_1) + \frac{1}{|g'(x_2)|} f(x_2) + \frac{1}{|g'(x_3)|} f(x_3)$$

$$= \sum_{x\in g^{-1}(g_0)} \frac{f(x)}{|g'(x)|}$$

How about the measure interpretation of $\delta(x)$?

This problematic actually.

$\int \delta(g(x)-g_0)\, f(x)\,dx$ is most naturally thought of

as

$$\int f(x)\,\overbrace{\left[\delta(g(x)-g_0)\,dx\right]}^{\text{pullback of the point measure at 0 under } g^{-1}}$$

but this pull back simply gives the measure of 1 to the set $\{x \mid g(x)=g_0\}$ and does not specify how that measure is divided between the point in $\{x \mid g(x)=g_0\}$.

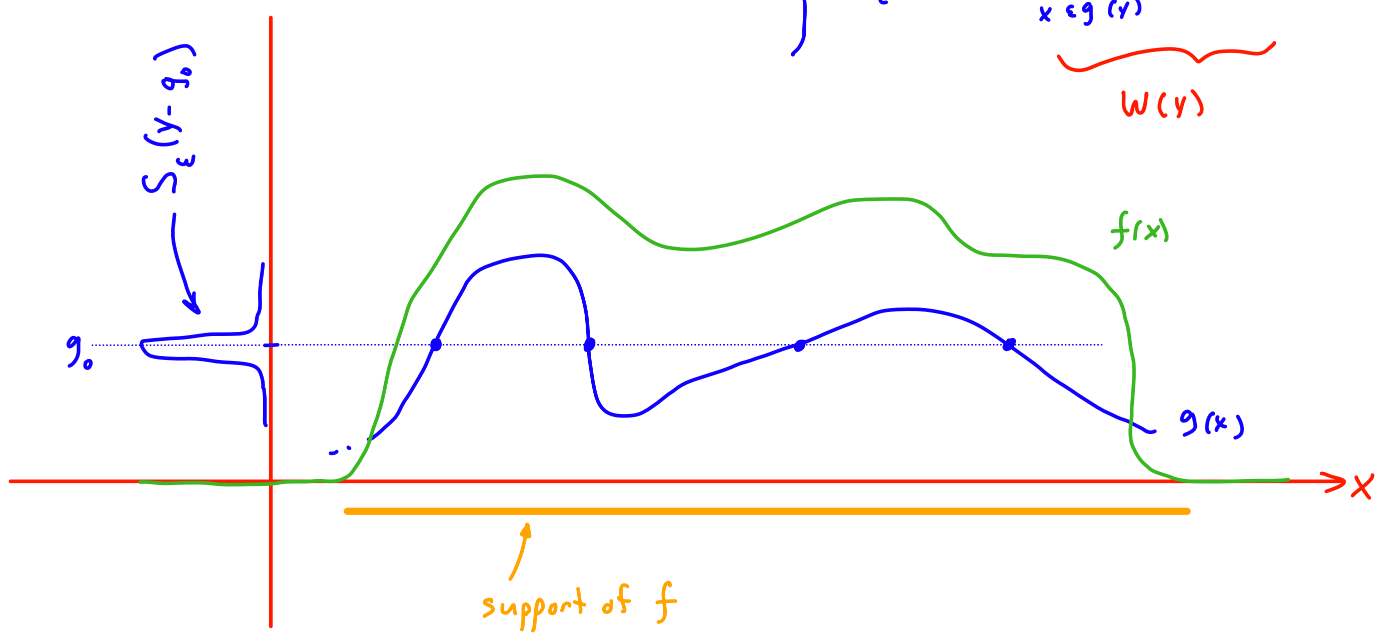
$$\int S_\varepsilon(g(x)-g_0)\, f(x)\,dx = \int \overbrace{S_\varepsilon(g(x)-g_0)\,\frac{f(x)}{|g'(x)|}}^{W(x)} \cdot |g'(x)|\,dx$$

now we apply the area formula with

$g:\mathbb{R}_x \to \mathbb{R}_y \qquad Jg \equiv |g'(x)|$

$$= \int \left(\sum_{x\in g^{-1}(y)} S_\varepsilon(g(x)-g_0)\,\frac{f(x)}{|g'(x)|} \right) dy$$

$$= \int S_\varepsilon(y-g_0) \underbrace{\sum_{x\in g^{-1}(y)} \left(f(x)/|g'(x)|\right)}_{W(y)} dy$$



as $\varepsilon \to 0$ the points where $S_\varepsilon(y-g_0)$ is big enough to worry about yeild an almost constant $W(y)$ approximately equal to $W(g_0) = \sum_{x\in g^{-1}(g_0)} f(x)/|g'(x)|$ and we get the formally expected result. The values g_0 where $g'(x)=0$ for some $x \in g^{-1}(g_0)$ have measure 0 in the range (by Sard's theorem)

So the integral (last one above) does not care about those g_0 even though you might ... i.e. the integral relation above is "not comenty" about that case.

<u>So</u>: • If S_ε has a support shrinks to the point g_0 and g_0 is a regular value of g (i.e. $g'(x) \neq 0$ $\forall x \in g^{-1}(g_0)$) then the result is true.

- If S_ε has asymptotic properties such that away from a region that is shrinks to g_0, we can show the integral goes to 0, we can again get the result

- The result is not true if g_0 is not a regular value of g.